Fock representations of the Lie superalgebra $q(n+1)$

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Abstract

For the Lie superalgebra $q(n+1)$ a description is given in terms of creation and annihilation operators, in such a way that the defining relations of $q(n+1)$ are determined by quadratic and triple supercommutation relations of these operators. Fock space representations V_p of $q(n+1)$ are defined by means of these creation and annihilation operators. These new representations are introduced as quotient modules of some induced module of $q(n+1)$. The representations V_p are not graded, but they possess a number of properties that are of importance for physical applications. For p a positive integer, these representations V_p are finite-dimensional, with a unique highest weight (of multiplicity 1). The Hermitian form that is consistent with the natural adjoint operation on $q(n+1)$ is shown to be positive definite on V_p. For $q(2)$ these representations are "dispin". For the general case of $q(n+1)$, many structural properties of V_p are derived.

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1 Introduction

Lie superalgebras and their irreducible representations (simple modules) have been the subject of much attention in both the mathematical [1, 2, 3] and the physics [4, 5, 6] literature. However, even for the simplest family of basic classical Lie superalgebras, namely $sl(m/n)$ or $gl(m/n)$, the understanding of all finite-dimensional simple modules has been a very difficult problem. The main reason for this difficulty has been the existence of so-called *atypical* modules [2]. Although partial progress was made in determining a character formula for atypical modules [7, 8, 9], the problem of determining the character for $gl(m/n)$ modules was solved only recently [10, 11] (see also [12] for a simpler algorithm).

The general linear Lie algebra $gl(n)$ has two super-analogues. The first is the Lie superalgebra $gl(n/1)$, for which the representations (even the atypical ones) are now well known : e.g., a Gelfand-Zetlin basis has been introduced and its transformations have been determined [13, 14], and representations of $sl(n/1)$ have been studied [15, 16]. The second super-analogue is the strange Lie superalgebra $q(n)$. Also this Lie superalgebra has received attention recently. In particular, characters of finite-dimensional irreducible graded representations of $q(n)$ have been determined [17, 18, 19], both in the typical and atypical case. In a different context, oscillator realizations have been given [20].

The finite-dimensional irreducible graded representations of $q(n)$ have the strange property that the multiplicity of the highest (or lowest) weight is in general greater than 1 [18]. From the physical point of view, where one wishes to interpret the representation space as a Hilbert space with a unique vacuum, this situation is not very favourable.

In the present paper, the purpose is to study a new class of irreducible finite-dimensional representations of the Lie superalgebra $q(n)$, which have certain properties that are required in a physical context. In particular, the highest weight has multiplicity 1 (so there is a unique highest weight vector, up to a factor), and the representation space can be naturally equipped with a symmetric and nondegenerate positive definite Hermitian form (inner product). Moreover, creation and annihilation generators (or operators) are introduced for $q(n)$ such that the representation space is a Fock space. The only property that has to be abandoned is the grading of the representation space (but from the physical point of view, this grading is no requirement).

The structure of the paper is as follows. In section 2 the main definitions are given concerning the Lie superalgebra $q(n+1)$. In section 3 a new basis for $q(n+1)$ is given in terms of so-called creation and annihilation operators. The new class of representations of $q(n+1)$ is introduced in section 4. These representations are defined by means of an

induced module $\bar{V}_p$; the actual irreducible representation V_p which is of interest is then a quotient module of $\bar{V}_p$. To have some idea about the structure of V_p, we first consider the low rank case of $q(2)$ in section 5. Here, the representation has a "dispin" structure. Section 6 goes back to the general case $q(n+1)$, and includes several (technical) properties concerning the structure of $\bar{V}_p$, paving the way to determining the structure of the simple modules V_p. This is performed in section 7, where in particular we give the dimension and character of V_p, and show that the Hermitian form is positive definite.

2 The Lie superalgebra $q(n+1)$

The Lie superalgebra $q(n+1)$ can be determined through its defining representation, i.e.

$$q(n+1) = \left\{ \left(\begin{array}{cc} A & B \\ B & A \end{array} \right) \mid A, B \in gl(n+1) \right\}, \tag{1}$$

where the matrices with $B = 0$ are even, or elements of $q(n+1)_{\bar{0}}$, and those with $A = 0$ odd, or elements of $q(n+1)_{\bar{1}}$. The subalgebra $sq(n+1)$ consists of those elements with $\mathrm{tr}(B) = 0$. The Lie superalgebras $q(n+1)$ and $sq(n+1)$ still contain a one-dimensional center $\mathbb{C}I$, where I is the identity matrix. Hence one defines the quotient Lie superalgebra $psq(n+1)$ as $sq(n+1)/\mathbb{C}I$. The notation $q(n+1)$, $sq(n+1)$ and $psq(n+1)$ is due to Penkov and others [17]; in the notation of Kac [1, 2] we have $\bar{Q}(n) = sq(n+1)$ and $Q(n) = psq(n+1)$. Recall that $Q(n)$ is a simple Lie superalgebra for $n \geq 2$, and that it is one of the series of classical (but "strange") Lie superalgebras in the classification of Kac [1]. For the development of representation theory, we shall be working mainly with $q(n+1)$.

Thus $q(n+1)$ can be defined as the Lie superalgebra with $(n+1)^2$ even basis elements $e_{ij}^{\bar{0}}$ $(i, j = 0, 1, \ldots, n)$ and $(n+1)^2$ odd basis elements $e_{ij}^{\bar{1}}$ $(i, j = 0, 1, \ldots, n)$, satisfying the bracket relation

$$[\![e_{ij}^{\sigma}, e_{kl}^{\theta}]\!] = \delta_{jk} e_{il}^{\sigma+\theta} - (-1)^{\sigma\theta} \delta_{il} e_{kj}^{\sigma+\theta}, \tag{2}$$

where $\sigma, \theta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, and $i, j, k, l \in \{0, 1, \ldots, n\}$. In this paper, we shall use $[\![\, ,]\!]$ for the Lie superalgebra bracket, and write explicitly $[\, ,]$ (resp. $\{\, , \}$) if this stands for a commutator (resp. anti-commutator). Let E_{ij} denote the $(n+1) \times (n+1)$ matrix with 1 in position (i, j) and 0 elsewhere (indices running from 0 to n), then the defining representation of $q(n+1)$ is given by the map

$$e_{ij}^{\bar{0}} \to \left(\begin{array}{cc} E_{ij} & 0 \\ 0 & E_{ij} \end{array} \right), \qquad e_{ij}^{\bar{1}} \to \left(\begin{array}{cc} 0 & E_{ij} \\ E_{ij} & 0 \end{array} \right) \tag{3}$$

onto matrices of order $2(n+1)$.

Following the definition of Cartan subalgebra as a maximal nilpotent subalgebra coinciding with its own normalizer, a Cartan subalgebra H of $G = q(n+1)$ is given by $H = H_{\bar{0}} \oplus H_{\bar{1}}$, where $H_{\bar{0}} = \text{span}\{e_{ii}^{\bar{0}} \mid i = 0, 1, \dots n\}$ and $H_{\bar{1}} = \text{span}\{e_{ii}^{\bar{1}} \mid i = 0, 1, \dots n\}$ [17, 21]. This subalgebra is not abelian, and since the elements of $H_{\bar{1}}$ are odd, the root generators $e_{ij}^{\bar{1}}$ are not eigenvectors of $H_{\bar{1}}$. Therefore, to give a root decomposition of G it is more convenient [21] to work with the abelian subalgebra $H_{\bar{0}}$. The dual space $H_{\bar{0}}^*$ has the standard basis $\{\epsilon_0, \epsilon_1, \dots, \epsilon_n\}$, in terms of which the roots of G can be described. The elements $e_{ij}^{\bar{0}}$ $(i \neq j)$ are even root vectors corresponding to the root $\epsilon_i - \epsilon_j$; the elements $e_{ij}^{\bar{1}}$ $(i \neq j)$ are odd root vectors also corresponding to the root $\epsilon_i - \epsilon_j$; the elements $e_{ii}^{\bar{1}}$ from $H_{\bar{1}}$ can then be interpreted as odd root vectors corresponding to the root 0. Note that every root $\epsilon_i - \epsilon_j$ $(i \neq j)$ has multiplicity 2 (counting once as even and once as odd root). Let, as usual, Δ be the set of all roots, $\Delta^{\bar{0}}$ (resp. $\Delta^{\bar{1}}$) be the set of even (resp. odd) roots :

$$\Delta^{\bar{0}} = \{\epsilon_i - \epsilon_j \mid 0 \leq i \neq j \leq n\}, \qquad \Delta^{\bar{1}} = \Delta^{\bar{0}} \cup \{0\}. \tag{4}$$

The positive roots are

$$\Delta_+ = \Delta_+^{\bar{0}} = \Delta_+^{\bar{1}} = \{\epsilon_i - \epsilon_j \mid 0 \leq i < j \leq n\}. \tag{5}$$

With this choice of positive roots the weights $\lambda = \sum_{i=0}^n \lambda_i \epsilon_i \in H_{\bar{0}}^*$ are partially ordered by $\lambda \leq \mu$ iff $\mu - \lambda = \sum k_\alpha \alpha$ where $\alpha \in \Delta_+$ and k_α are nonnegative integers. The adjoint representation has $\epsilon_0 - \epsilon_n$ as highest weight, with multiplicity 2. The defining representation has highest weight ϵ_0, also with multiplicity 2.

Let V be a linear space over $\mathbb{C}$, and denote by $gl(V)$ the space of endomorphisms of V. A representation ρ is a linear mapping from G to $gl(V)$ such that

$$\rho([\![x, y]\!]) = \rho(x)\rho(y) - (-1)^{\sigma\theta}\rho(y)\rho(x), \qquad \forall x \in G_\sigma, y \in G_\theta;\ \sigma, \theta \in \mathbb{Z}_2. \tag{6}$$

Then V is a G-module with $xv = \rho(x)v$ for $x \in G$ and $v \in V$. If, moreover, V is a $\mathbb{Z}_2$-graded linear space, i.e. $V = V_{\bar{0}} \oplus V_{\bar{1}}$, then also $gl(V)$ is naturally graded, $gl(V) = gl(V)_{\bar{0}} \oplus gl(V)_{\bar{1}}$, and then ρ is a graded representation (and V a graded G-module) if $\rho(x) \in gl(V)_\sigma$ for $x \in G_\sigma$. For Lie superalgebras, one usually considers only the graded modules when studying representation theory [1, 3]. Here, we shall see that $q(n+1)$ has a class of interesting non-graded modules.

Graded modules of $q(n+1)$ were considered by Penkov and Serganova [17, 18, 19]. In particular, they showed that the finite-dimensional irreducible representations V of $q(n+1)$ are characterized by a highest weight $\lambda = \sum_i \lambda_i \epsilon_i$, such that $\lambda_i - \lambda_{i+1}$ is a nonnegative integer and $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = \lambda_{i+1} = 0$. The dimension of the highest

weight space V_λ ($\lambda \neq 0$) is given by [18, page 150]

$$\dim(V_\lambda) = 2^{1+[(\#\lambda-1)/2]}, \tag{7}$$

where $\#\lambda$ is the number of nonzero coordinates λ_i, and $[t]$ is the integer part of t. For example, for the defining representations with $\lambda = (1, 0, \dots, 0)$ and the adjoint representation with $\lambda = (1, 0, \dots, 0, -1)$, the highest weight space has dimension 2.

From the physical point of view, it is unusual to have a highest (or lowest) weight with multiplicity greater than 1, since this is normally associated to a "unique vacuum".

In the present paper we shall show that $q(n+1)$ has a class of interesting non-graded representations, with a unique highest weight vector (i.e. highest weight multiplicity 1), and which are also finite-dimensional. Moreover, these representations can be interpreted as Hermitian Fock representations generated by n even and n odd creation operators.

3 Creation and annihilation operators for $q(n+1)$

Let $a_i^\xi(\sigma)$ be the following elements of $q(n+1)$:

$$a_i^+(\sigma) = e_{i,0}^\sigma, \qquad a_i^-(\sigma) = e_{0,i}^\sigma, \qquad \sigma \in \mathbb{Z}_2, \quad i \in \{1, \dots, n\}. \tag{8}$$

It is easy to verify that these operators satisfy the following relations :

$$[\![a_i^-(\sigma), a_j^-(\theta)]\!] = [\![a_i^+(\sigma), a_j^+(\theta)]\!] = 0, \tag{9}$$

$$[\![[\![a_i^+(\sigma), a_j^-(\theta)]\!], a_k^+(\omega)]\!] =$$
$$\delta_{jk} a_i^+(\sigma+\theta+\omega) + (-1)^{\sigma\theta+\theta\omega+\omega\sigma} \delta_{ij} a_k^+(\sigma+\theta+\omega); \tag{10}$$

$$[\![[\![a_i^+(\sigma), a_j^-(\theta)]\!], a_k^-(\omega)]\!] =$$
$$-(-1)^{\sigma\theta} \delta_{ij} a_k^-(\sigma+\theta+\omega) - (-1)^{\theta\omega+\omega\sigma} \delta_{ik} a_j^-(\sigma+\theta+\omega), \tag{11}$$

where $\sigma, \theta, \omega \in \mathbb{Z}_2$, $i, j, k \in \{1, \dots, n\}$. It is convenient to introduce the following notational difference between the even and odd operators :

$$b_i^\pm = a_i^\pm(\bar{0}), \qquad f_i^\pm = a_i^\pm(\bar{1}). \tag{12}$$

The operators b_i^+, f_i^+ (resp. b_i^-, f_i^-) shall be referred to as creation (resp. annihilation) operators for the Lie superalgebra $q(n+1)$ (even though they generate only the subalgebra $sq(n+1)$).

A definition of creation and annihilation operators (or generators) of a simple Lie (super)algebra $\mathcal{L}$ and of the related Fock spaces was given in [22, §2]. The motivation for

introducing such operators stems from the observation that the creation and annihilation operators (CAO's) of certain algebras have a direct physical significance. We have in mind the para-Fermi and the para-Bose operators, which generalize the statistics of spinor and tensor fields in quantum field theory [23]. Any n pairs of parafermions are CAO's of the orthogonal Lie algebra $so(2n+1) \equiv B_n$ [24, 25]. The parabosons do not generate a Lie algebra, they generate a Lie superalgebra [26]. Any n pairs of them are CAO's of the orthosymplectic Lie superalgebra $osp(1,2n) \equiv B(0,n)$ [27].

In [28] the question was raised whether each simple Lie (super)algebra can be generated by creation and annihilation generators. The answer is positive for all algebras from the classes A, B, C and D of simple Lie algebras [29] and for some Lie superalgebras. So far however only the CAO's and the Fock representations of $sl(n+1)$ (A-statistics) [29, 30] and of Lie superalgebras $sl(1/n)$ ($A-$superstatistics) [22] were studied in somewhat greater detail. The present paper is another contribution along this line for the Lie superalgebra $q(n+1)$.

Coming back to the CAO's (8) we note that the operators $b_i^{\pm}$ satisfy the relations of A-statistics [30], whereas the operators $f_i^{\pm}$ satisfy the relations of A-superstatistics [22]. Here, we shall refer to the combined relations (9)-(11) as Q-statistics. Clearly, the linear envelope of

$$\{a_i^{\xi}(\sigma), [\![a_i^{\xi}(\sigma), a_j^{\eta}(\theta)]\!] | \xi,\eta \in \{+,-\},\ \sigma,\theta \in \mathbb{Z}_2,\ i,j \in \{1,\ldots,n\}\} \tag{13}$$

is equal to the Lie superalgebra $sq(n+1)$.

4 Fock space for $q(n+1)$

We shall define a Fock space for $G = q(n+1)$ using an induced module. First of all, from the commutation relations of $q(n+1)$ it is straightforward to see that $q(1) = \mathrm{span}\{e_{00}^{\bar{0}}, e_{00}^{\bar{1}}\}$ and $q(n) = \mathrm{span}\{e_{i,j}^{\sigma} \mid i,j = 1,2,\ldots,n;\ \sigma = \bar{0},\bar{1}\}$ are subalgebras of $q(n+1)$ with $[\![q(1), q(n)]\!] = 0$. So consider the subalgebra

$$\tilde{G} = q(1) \oplus q(n). \tag{14}$$

Let

$$\begin{aligned} P &= \mathrm{span}\{b_1^-, \ldots, b_n^-, f_1^-, \ldots, f_n^-\}, \\ N &= \mathrm{span}\{b_1^+, \ldots, b_n^+, f_1^+, \ldots, f_n^+\}; \end{aligned} \tag{15}$$

these are two abelian subalgebras of G. Then $G = \tilde{G} + P + N$, where the sum is direct as linear spaces. Since $[\![\tilde{G}, P]\!] = P$, $\tilde{G} + P$ is also a subalgebra of G.

The Lie superalgebra $q(1)$ has basis elements $e_{00}^{\bar{0}}$ and $e_{00}^{\bar{1}}$ with supercommutation relations

$$[e_{00}^{\bar{0}}, e_{00}^{\bar{0}}] = 0, \qquad [e_{00}^{\bar{0}}, e_{00}^{\bar{1}}] = 0, \qquad \{e_{00}^{\bar{1}}, e_{00}^{\bar{1}}\} = 2e_{00}^{\bar{0}}. \tag{16}$$

Clearly, this Lie superalgebra has one-dimensional irreducible modules $\mathbb{C}v_0$ characterized by a number p, with action

$$e_{00}^{\bar{0}} v_0 = p\, v_0, \qquad e_{00}^{\bar{1}} v_0 = \sqrt{p}\, v_0. \tag{17}$$

In principle p can be any complex number, but later we shall be interested only in the case that p is a positive real number. The $q(1)$-module $\mathbb{C}v_0$ can be extended to a $\tilde{G} = q(1) \oplus q(n)$-module by letting $xv_0 = 0$ for all $x \in q(n)$. Requiring that $xv_0 = 0$ for every $x \in P$ it becomes a $(\tilde{G} + P)$-module.

We now define the following induced G-module :

$$\bar{V}_p = \mathrm{Ind}_{\tilde{G}+P}^{G} \mathbb{C}v_0 \cong U(G) \otimes_{\tilde{G}+P} \mathbb{C}v_0. \tag{18}$$

By the Poincaré-Birkhoff-Witt theorem for Lie superalgebras, we have

$$\bar{V}_p \cong U(N) \otimes \mathbb{C}v_0. \tag{19}$$

Thus a basis of $\bar{V}_p$ is given by the elements

$$\begin{aligned} &|p; k_1, l_1, k_2, l_2, \ldots, k_n, l_n\rangle = (b_1^+)^{k_1} (f_1^+)^{l_1} (b_2^+)^{k_2} (f_2^+)^{l_2} \cdots (b_n^+)^{k_n} (f_n^+)^{l_n} v_0, \\ &\quad l_i \in \{0, 1\},\ k_i = 0, 1, 2, \ldots. \end{aligned} \tag{20}$$

What are the identities that hold in this representation space $\bar{V}_p$? First of all, note that $\bar{V}_p$ has a unique highest weight equal to $p\epsilon_0$, corresponding to the unique (up to a factor) highest weight vector v_0. So the highest weight has multiplicity 1. On the other hand, (17) shows that v_0 is not an even nor an odd vector, i.e. the G-module $\bar{V}_p$ is not graded. Note that the weight of (20) is given by

$$p\epsilon_0 + \sum_{i=1}^{n} (k_i + l_i)(\epsilon_i - \epsilon_0). \tag{21}$$

Secondly, the vector v_0 can genuinely be called a "vacuum vector" since it satisfies

$$b_i^- v_0 = f_i^- v_0 = 0, \qquad (i = 1, 2, \ldots, n). \tag{22}$$

Furthermore, the following relations are valid :

$$\begin{aligned} &b_i^- b_j^+ v_0 = \delta_{ij} p\, v_0, \qquad f_i^- f_j^+ v_0 = \delta_{ij} p\, v_0, \\ &f_i^- b_j^+ v_0 = \delta_{ij} \sqrt{p}\, v_0, \qquad b_i^- f_j^+ v_0 = \delta_{ij} \sqrt{p}\, v_0. \end{aligned} \tag{23}$$

Note that relations (22) and (23) are also sufficient to define the representation $\bar{V}_p$.

In order to call $\bar{V}_p$ a Fock space, one further condition should be satisfied, namely it should be a Hilbert space consistent with the adjoint operation [30, 22]

$$(b_i^{\pm})^{\dagger} = b_i^{\mp}, \qquad (f_i^{\pm})^{\dagger} = f_i^{\mp}. \tag{24}$$

Thus we define a Hermitian form on $\bar{V}_p$ by

$$\langle v_0|v_0\rangle = 1,\ \langle b_i^+ v|w\rangle = \langle v|b_i^- w\rangle,\ \langle f_i^+ v|w\rangle = \langle v|f_i^- w\rangle, \qquad v, w \in \bar{V}_p. \tag{25}$$

In general $\bar{V}_p$ is not a Hilbert space. However, we shall see that if p is a positive integer, $\bar{V}_p$ has a quotient space which is a Hilbert space. Indeed, if p is a positive integer, the space $\bar{V}_p$ is shown to have a maximal submodule M_p. Then the quotient module $V_p = \bar{V}_p/M_p$ is an irreducible G-module. The Hermitian form is zero on M_p and on V_p it induces a positive definite metric. Thus V_p can genuinely be called a Fock space representation of $q(n+1)$.

The Lie superalgebra $q(n+1)$ contains a one-dimensional center,

$$I = \sum_{i=0}^{n} e_{ii}^{\bar{0}}. \tag{26}$$

Since $Iv_0 = pv_0$, and $\bar{V}_p$ (or V_p) is generated by v_0, it follows that $Iv = pv$ for every v in $\bar{V}_p$ (or V_p).

As we shall see, the structure of V_p or $\bar{V}_p$ is far from trivial. Before turning to the general case, let us first consider the low rank case of the Lie superalgebra $G = q(2)$.

5 Fock space for $q(2)$

Since $n = 1$ there is only one index for the creation and annihilation operators, so we shall simply denote $b_1^{\pm}, f_1^{\pm}$ by $b^{\pm}, f^{\pm}$.

The representation space $\bar{V}_p$ is spanned by the following vectors (notation of (20)) :

$$\begin{aligned} v_k &= |p;k,0\rangle = (b^+)^k v_0,\ k = 0, 1, \ldots; \\ w_k &= |p;k-1,1\rangle = (b^+)^{k-1} f^+ v_0,\ k = 1, 2, \ldots. \end{aligned} \tag{27}$$

The following actions of the annihilation operators on v_k and w_k can be computed using the triple relations (9) and (10), (11) :

$$b^- v_k = k(p-k+1)v_{k-1}, \tag{28}$$

$$f^- v_k = k\sqrt{p}\, v_{k-1} - k(k-1)w_{k-1}, \tag{29}$$
$$b^- w_k = \sqrt{p}\, v_{k-1} + (k-1)(p-k)w_{k-1}, \tag{30}$$
$$f^- w_k = pv_{k-1} - (k-1)\sqrt{p}\, w_{k-1}. \tag{31}$$

It is not difficult to verify the following, using the earlier defined metric on $\bar{V}_p$ and (28) :

$$\langle v_k | v_k \rangle = k!p(p-1)\cdots(p-k+1). \tag{32}$$

For fixed p, this expression can take positive and negative values, depending upon the value of k. Hence, $\bar{V}_p$ itself is not a Hilbert space representation. Next, let us investigate whether $\bar{V}_p$ is irreducible. Using the relations (25), (28)-(31), and induction, one can show that

$$\langle v_k | v_l \rangle = \delta_{kl} k!p(p-1)\cdots(p-k+1), \tag{33}$$
$$\langle w_k | w_l \rangle = \delta_{kl} (k-1)!p(p-1)\cdots(p-k+1), \tag{34}$$
$$\langle v_k | w_l \rangle = \delta_{kl} k!p(p-1)\cdots(p-k+1)/\sqrt{p}. \tag{35}$$

Now (33) and (34) imply that

$$\begin{aligned} (b^-)^k v_k &= k!p(p-1)\cdots(p-k+1)\, v_0, \\ (b^-)^{k-1} f^- w_k &= (k-1)!p(p-1)\cdots(p-k+1)\, v_0. \end{aligned} \tag{36}$$

Thus if p is not a positive integer these coefficients are not zero, implying that the vectors v_k and w_k cannot belong to a submodule of $\bar{V}_p$ (apart from the trivial submodule $\bar{V}_p$). In other words, if p is not a positive integer, $\bar{V}_p$ is irreducible.

Let us now consider the interesting case that p is a positive integer. Then $\bar{V}_p$ has a maximal submodule M_p. Since $\bar{V}_p$ is a module generated by a highest weight vector, the submodule is generated by primitive weight vectors, so let us determine when a weight vector $v_k + \beta w_k$ is primitive. The conditions $b^-(v_k + \beta w_k) = 0$ and $f^-(v_k + \beta w_k) = 0$ lead to one solution only, namely $k = p$ and $\beta = -\sqrt{p}$. Thus $v_p - \sqrt{p}\, w_p$ is a primitive vector generating the submodule M_p. The quotient module $V_p = \bar{V}_p / M_p$ is therefore a finite-dimensional module. A set of basis vectors of V_p, together with the corresponding weight, is given by

$$\begin{array}{ll} v_0 & p\epsilon_0 \\ v_1, w_1 & (p-1)\epsilon_0 + \epsilon_1 \\ v_2, w_2 & (p-2)\epsilon_0 + 2\epsilon_1 \\ \vdots & \vdots \\ v_{p-1}, w_{p-1} & \epsilon_0 + (p-1)\epsilon_1 \\ v_p + \sqrt{p}\, w_p & p\epsilon_1. \end{array} \tag{37}$$

The top and bottom weight appear with multiplicity 1, the other weights have multiplicity 2. From the weight structure one can determine the decomposition of this finite-dimensional $q(2)$ module with respect to the subalgebra $gl(2) \subset q(2)$:

$$V_p \to (p,0) \oplus (p-1,1), \qquad\qquad (p>1). \tag{38}$$

So V_p splits into two irreducible $gl(2)$ modules, both of which have been labelled by their highest weight (in the (ϵ_0,ϵ_1)-basis). So for $p>1$ these $q(2)$ representations could be referred to as "dispin" representations, similar to the known dispin representations of $osp(1,2)$ [31, 32]. For $p=1$, the decomposition is simply $V_p \to (p,0)$. The dimension follows easily :

$$\dim V_p = 2p. \tag{39}$$

It is possible to give an orthonormal basis for V_p, in terms of the above basis vectors v_k, w_k. Since

$$\langle v_k \pm \sqrt{k} w_k | v_k \pm \sqrt{k} w_k \rangle = 2(1 \pm \sqrt{k/p}) k! p!/(p-k)!, \tag{40}$$

$$\langle v_k + \sqrt{k} w_k | v_k - \sqrt{k} w_k \rangle = 0, \tag{41}$$

one can define

$$\phi_k = \left(\frac{(p-k)!}{2k!p!(1+\sqrt{k/p})} \right)^{1/2} (v_k + \sqrt{k} w_k),\ (k=1,\ldots,p) \tag{42}$$

$$\psi_k = \left(\frac{(p-k)!}{2k!p!(1-\sqrt{k/p})} \right)^{1/2} (v_k - \sqrt{k} w_k),\ (k=0,\ldots,p-1). \tag{43}$$

These vectors are orthonormal :

$$\langle \phi_k | \phi_l \rangle = \langle \psi_k | \psi_l \rangle = \delta_{kl}, \qquad \langle \phi_k | \psi_l \rangle = 0. \tag{44}$$

The action of the creation and annihilation operators on this basis can be computed. We have :

$$\begin{aligned} f^+\phi_k &= \frac{1}{2}\left[(\sqrt{p}-\sqrt{k})(\sqrt{p}+\sqrt{k+1}) \right]^{1/2} \phi_{k+1} \\ &\quad -\frac{1}{2}\left[(\sqrt{p}-\sqrt{k})(\sqrt{p}-\sqrt{k+1}) \right]^{1/2} \psi_{k+1} \end{aligned} \tag{45}$$

$$\begin{aligned} f^+\psi_k &= \frac{1}{2}\left[(\sqrt{p}+\sqrt{k})(\sqrt{p}+\sqrt{k+1}) \right]^{1/2} \phi_{k+1} \\ &\quad -\frac{1}{2}\left[(\sqrt{p}+\sqrt{k})(\sqrt{p}-\sqrt{k+1}) \right]^{1/2} \psi_{k+1} \end{aligned} \tag{46}$$

$$b^+\phi_k = \frac{1}{2}(\sqrt{k+1}+\sqrt{k}) \left[(\sqrt{p}-\sqrt{k})(\sqrt{p}+\sqrt{k+1}) \right]^{1/2} \phi_{k+1}$$

$$+\frac{1}{2}(\sqrt{k+1}-\sqrt{k})\left[(\sqrt{p}-\sqrt{k})(\sqrt{p}-\sqrt{k+1})\right]^{1/2}\psi_{k+1} \quad (47)$$

$$b^+\psi_k = \frac{1}{2}(\sqrt{k+1}-\sqrt{k})\left[(\sqrt{p}+\sqrt{k})(\sqrt{p}+\sqrt{k+1})\right]^{1/2}\phi_{k+1} + \frac{1}{2}(\sqrt{k+1}+\sqrt{k})\left[(\sqrt{p}+\sqrt{k})(\sqrt{p}-\sqrt{k+1})\right]^{1/2}\psi_{k+1}. \quad (48)$$

The action of the annihilation operators follows immediately from $b^- = (b^+)^\dagger$, $f^- = (f^+)^\dagger$. For example,

$$f^-\phi_k = \frac{1}{2}\left[(\sqrt{p}-\sqrt{k-1})(\sqrt{p}+\sqrt{k})\right]^{1/2}\phi_{k-1} + \frac{1}{2}\left[(\sqrt{p}+\sqrt{k-1})(\sqrt{p}+\sqrt{k})\right]^{1/2}\psi_{k-1}. \quad (49)$$

6 Structure of the module $\bar{V}_p$

In this section we return to the general case $q(n+1)$. By calculating the action of creation and annihilation operators on basis vectors of the induced module $\bar{V}_p$, the way is prepared to determine the structure of the irreducible quotient module V_p of $\bar{V}_p$.

Recall that a basis for $\bar{V}_p$ is given by the vectors

$$|p;\boldsymbol{k},\boldsymbol{l}\rangle = |p;k_1,l_1,k_2,l_2,\ldots,k_n,l_n\rangle = (b_1^+)^{k_1}(f_1^+)^{l_1}(b_2^+)^{k_2}(f_2^+)^{l_2}\cdots(b_n^+)^{k_n}(f_n^+)^{l_n}v_0,$$
$$l_i \in \{0,1\},\ k_i = 0,1,2,\ldots. \quad (50)$$

Since all creation operators b_i^+ and f_i^+ supercommute, a different order of the creation operators in (50) can produce only a sign change.

In the standard basis, the weight of the vector $|p;\boldsymbol{k},\boldsymbol{l}\rangle$ is given by

$$\mathrm{weight}(|p;\boldsymbol{k},\boldsymbol{l}\rangle) = p\epsilon_0 + \sum_{i=1}^n (k_i+l_i)(\epsilon_i-\epsilon_0) = \left(p-\sum_{i=1}^n (k_i+l_i), k_1+l_1,\ldots,k_n+l_n\right). \quad (51)$$

Thus every weight of $\bar{V}_p$ is of the form

$$\lambda_m = (p-\sum_{i=1}^n m_i, m_1,\ldots,m_n), \qquad m_i = 0,1,2,\ldots; \quad (52)$$

conversely, every weight of the form (52) is a weight of $\bar{V}_p$. Since the l_i in (50) are either 0 or 1, it follows that the multiplicity of the weight λ_m in $\bar{V}_p$ is given by

$$\mathrm{mult}_{\bar{V}_p}(p-\sum_{i=1}^n m_i, m_1,\ldots,m_n) = 2^{\gamma(m_1)+\cdots+\gamma(m_n)}, \quad (53)$$

where $\gamma(m_i) = 0$ if $m_i = 0$ and $\gamma(m_i) = 1$ if $m_i \neq 0$.

The action of the creation operators on the basis of $\bar{V}_p$ is very simple. It will be convenient to denote the basis vectors in the right hand side of such actions only by means of the labels that are effectively changed by the action. So, instead of writing

$$b_j^+|p;\boldsymbol{k},\boldsymbol{l}\rangle = |p;k_1,l_1,\ldots,k_j+1,l_j,\ldots,k_n,l_n\rangle,$$
$$f_j^+|p;\boldsymbol{k},\boldsymbol{l}\rangle = \delta_{l_j,0}(-1)^{l_1+\cdots+l_{j-1}}|p;k_1,l_1,\ldots,k_j,l_j+1,\ldots,k_n,l_n\rangle,$$

we abbreviate this to :

$$b_j^+|p;\boldsymbol{k},\boldsymbol{l}\rangle = |k_j+1\rangle, \tag{54}$$
$$f_j^+|p;\boldsymbol{k},\boldsymbol{l}\rangle = \delta_{l_j,0}(-1)^{l_1+\cdots+l_{j-1}}|l_j+1\rangle. \tag{55}$$

The action of the annihilation operators is more complicated, and here the notational convention just introduced will be very useful.

Proposition 1 *The action of the annihilation operators in the module $\bar{V}_p$ is given by :*

$$\begin{aligned} f_j^-|p;\boldsymbol{k},\boldsymbol{l}\rangle &= (-1)^{l_1+\cdots+l_{j-1}} l_j(p+1+k_j-\sum_{i=1}^n(k_i+l_i))|l_j-1\rangle \\ &\quad +(-1)^{l_1+\cdots+l_n}k_j\sqrt{p}|k_j-1\rangle - (-1)^{l_1+\cdots+l_{j-1}}\delta_{l_j,0}k_j(k_j-1)|k_j-2,l_j+1\rangle \\ &\quad -\sum_{\substack{i=1\\ i\neq j}}^n(-1)^{l_1+\cdots+l_{i-1}}\delta_{l_i,0}k_ik_j|k_j-1,k_i-1,l_i+1\rangle \\ &\quad +\sum_{\substack{i=1\\ i\neq j}}^n(-1)^{l_1+\cdots+l_{i-1}}l_ik_j|k_j-1,k_i+1,l_i-1\rangle; \end{aligned} \tag{56}$$

$$\begin{aligned} (-1)^{l_{j+1}+\cdots+l_n}b_j^-|p;\boldsymbol{k},\boldsymbol{l}\rangle &= (-1)^{l_{j+1}+\cdots+l_n}k_j(p+1-l_j-\sum_{i=1}^n(k_i+l_i))|l_j-1\rangle \\ &\quad +l_j\sqrt{p}|l_j-1\rangle + \sum_{\substack{i=1\\ i\neq j}}^n(-1)^{l_i+\cdots+l_n}\theta_{ij}\delta_{l_i,0}k_il_j|l_j-1,k_i-1,l_i+1\rangle \\ &\quad -\sum_{\substack{i=1\\ i\neq j}}^n(-1)^{l_i+\cdots+l_n}\theta_{ij}l_il_j|l_j-1,k_i+1,l_i-1\rangle, \end{aligned} \tag{57}$$

where $\theta_{ij} = +1$ if $i<j$ and $\theta_{ij} = -1$ if $i>j$.

Proof. We shall sketch the proof for the action of f_j^-; that for b_j^- is similar. The proof uses induction on n.

As a first step, the action of f_1^+ will be determined. Denote $y_i = (b_i^+)^{k_i}(f_i^+)^{l_i}$. Then,

$$f_1^-|p;\boldsymbol{k},\boldsymbol{l}\rangle = f_1^- y_1 y_2 \cdots y_n v_0 = [\![f_1^-, y_1]\!] y_2 \cdots y_n v_0 + (-1)^{l_1} y_1 f_1^- y_2 \cdots y_n v_0. \tag{58}$$

Since the weight of $f_1^- y_2 \cdots y_n v_0$, which is $(p+1-\sum_{i=2}^n (k_i+l_i), -1, k_2+l_2, \ldots, k_n+l_n)$, is not of the form (52), the vector cannot belong to $\bar{V}_p$, so the second term in (58) has to be zero. Using $[\![f_1^-, b_1^+]\!] = e_{00}^{\bar{1}} - e_{11}^{\bar{1}}$ and $[\![f_1^-, f_1^+]\!] = e_{00}^{\bar{0}} + e_{11}^{\bar{0}}$, one finds

$$[\![f_1, y_1]\!] = \sum_{r=1}^{k_1-1} (b_1^+)^r (e_{00}^{\bar{1}} - e_{11}^{\bar{1}})(b_1^+)^{k_1-r-1}(f_1^+)^{l_1} + l_1 (b_1^+)^{k_1}(e_{00}^{\bar{0}} + e_{11}^{\bar{0}}). \tag{59}$$

From the weight of y_i and v_0 one obtains

$$(e_{00}^{\bar{0}} + e_{11}^{\bar{0}}) y_2 \cdots y_n v_0 = (p - k_2 - l_2 - \cdots - k_n - l_n) y_2 \cdots y_n v_0. \tag{60}$$

Next we need to determine the action of $e_{00}^{\bar{1}}$ and $e_{11}^{\bar{1}}$ on vectors of the form $y_1' y_2 \cdots y_n v_0$, where $y_1' = (b_1^+)^{k_1'}(f_1^+)^{l_1}$ with $k_1' = k_1 - r - 1$:

$$\begin{aligned} e_{00}^{\bar{1}} y_1' y_2 \cdots y_n v_0 &= [\![e_{00}^{\bar{1}}, y_1' y_2 \cdots y_n]\!] v_0 + (-1)^{l_1+\cdots l_n} y_1' y_2 \cdots y_n \sqrt{p}\, v_0 \\ &= [\![e_{00}^{\bar{1}}, y_1']\!] y_2 \cdots y_n v_0 + \sum_{i=2}^n y_1' \cdots y_{i-1} [\![e_{00}^{\bar{1}}, y_i]\!] y_{i+1} \cdots y_n v_0 \\ &\qquad + (-1)^{l_1+\cdots l_n} \sqrt{p}\, y_1' y_2 \cdots y_n v_0. \end{aligned}$$

Every term in this expression can be determined explicitly using

$$[\![e_{00}^{\bar{1}}, y_i]\!] = -\delta_{l_i,0} k_i (b_i^+)^{k_i-1} f_i^+ + l_i (b_i^+)^{k_i+1}, \tag{61}$$

which follows from the supercommutator of $e_{00}^{\bar{1}}$ with b_i^+ and f_i^+. Similarly, for the action of $e_{11}^{\bar{1}}$ one finds

$$e_{11}^{\bar{1}} y_1' y_2 \cdots y_n v_0 = [\![e_{11}^{\bar{1}}, y_1']\!] y_2 \cdots y_n v_0, \tag{62}$$

since $[\![e_{11}^{\bar{1}}, y_i]\!] = 0$ for $i > 1$ and $e_{11}^{\bar{1}} v_0 = 0$. Using the supercommutator of $e_{11}^{\bar{1}}$ with b_1^+ and f_1^+, there comes

$$[\![e_{11}^{\bar{1}}, y_1']\!] = \delta_{l_1,0} k_1' (b_1^+)^{k_1'-1} f_1^+ + l_1 (b_1^+)^{k_1'+1}. \tag{63}$$

Collecting now all contributions yields the action of f_1^+ on $|p;\boldsymbol{k},\boldsymbol{l}\rangle$, as given in the proposition.

In the second step, we use induction in the following way. First observe that for $j = 2$,

$$f_2^- y_1 y_2 \cdots y_n v_0 = [\![f_2^-, y_1]\!] y_2 \cdots y_n v_0 + (-1)^{l_1} y_1 f_2^- y_2 \cdots y_n v_0. \tag{64}$$

But the action $f_2^- y_2 \cdots y_n v_0$ is formally the same as the (known) action $f_1^- y_1 y_2 \cdots y_n v_0$, by relabelling of indices. More generally, for $j \geq 2$,

$$f_j^- y_1 y_2 \cdots y_n v_0 = [\![f_j^-, y_1]\!] y_2 \cdots y_n v_0 + (-1)^{l_1} y_1 f_j^- y_2 \cdots y_n v_0. \tag{65}$$

Once again, $f_j^- y_2 \cdots y_n v_0$ can formally be reduced to $f_{j-1}^- y_1 \cdots y_{n-1} v_0$, on which one uses the induction hypothesis. So what remains to be determined is the first term in (65). Since $[f_j^-, b_1^+]$ commutes with b_1^+ (see (10)), one finds

$$[\![f_j^-, (b_1^+)^{k_1} (f_1^+)^{l_1}]\!] = (-1)^{l_1} k_1 (b_1^+)^{k_1-1} (f_1^+)^{l_1} [f_j^-, b_1^+] + l_1 (b_1^+)^{k_1} \{f_j^-, f_1^+\}. \tag{66}$$

To calculate $[f_j^-, b_1^+] y_2 \cdots y_n v_0$, one uses again the triple relations (10) and (11) :

$$[f_j^-, b_1^+] y_2 \cdots y_n v_0 = (-1)^{l_2 + \cdots + l_{j-1}} y_2 \cdots y_{j-1} [\![[f_j^-, b_1^+], y_j]\!] y_{j+1} \cdots y_n v_0; \tag{67}$$

furthermore

$$[\![[f_j^-, b_1^+], y_j]\!] = -k_j f_1^+ (b_j^+)^{k_j - 1} (f_j^+)^{l_j} - l_j (b_j^+)^{k_j} b_1^+, \tag{68}$$

where again the triple relations have been used. In a similar way, the action $\{f_j^-, f_1^+\} y_2 \cdots y_n v_0$ is determined. Together they yield the first term in the right hand side of (65). Combining then the coefficients of all identical vectors in the right hand side of (65) proves the proposition. □

Now we wish to determine the vectors in $\bar{V}_p$ that are annihilated by all annihilation operators, i.e. by

$$P = \mathrm{span}\{b_1^-, \ldots, b_n^-, f_1^-, \ldots, f_n^-\}. \tag{69}$$

For this purpose, and inspired by the right hand sides in (57) and (56), we introduce the following weight vectors (still using the notational convention introduced in (54)-(55)) :

$$\begin{aligned} X(p; \boldsymbol{k}, \boldsymbol{l}) \ &= \ (-1)^{l_1 + \cdots + l_n} \sqrt{p} |p; \boldsymbol{k}, \boldsymbol{l}\rangle - \sum_{i=1}^n (-1)^{l_1 + \cdots + l_i} l_i |k_i + 1, l_i - 1\rangle \\ &\quad - \sum_{i=1}^n (-1)^{l_1 + \cdots + l_i} \delta_{l_i, 0} k_i |k_i - 1, l_i + 1\rangle. \end{aligned} \tag{70}$$

Then we have :

Proposition 2 *For the weight vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$ and $X(p; \boldsymbol{k}, \boldsymbol{l})$ the following equalities hold (again we use the convention that in the right hand side only the labels k_i and l_i that are effectively changed are withheld) :*

$$b_j^- |p; \boldsymbol{k}, \boldsymbol{l}\rangle \ = \ k_j (p + 1 - \sum_{i=1}^n (k_i + l_i)) |k_j - 1\rangle + (-1)^{l_1 + \cdots + l_{j-1}} l_j X(p; l_j - 1), \tag{71}$$

$$f_j^-|p;\boldsymbol{k},\boldsymbol{l}\rangle = (-1)^{l_1+\cdots+l_{j-1}} l_j(p+1-\sum_{i=1}^n (k_i+l_i))|l_j-1\rangle + k_j X(p;k_j-1), \quad (72)$$

$$b_j^+ X(p;\boldsymbol{k},\boldsymbol{l}) = X(p;k_j+1) + (-1)^{l_1+\cdots+l_j}\delta_{l_j,0}|l_j+1\rangle, \quad (73)$$

$$f_j^+ X(p;\boldsymbol{k},\boldsymbol{l}) = -\delta_{l_j,0}(-1)^{l_1+\cdots+l_{j-1}} X(p;l_j+1) + |k_j+1\rangle, \quad (74)$$

$$b_j^- X(p;\boldsymbol{k},\boldsymbol{l}) = k_j(p-\sum_{i=1}^n (k_i+l_i))X(p;k_j-1), \quad (75)$$

$$f_j^- X(p;\boldsymbol{k},\boldsymbol{l}) = (-1)^{l_1+\cdots+l_{j-1}} l_j(p-\sum_{i=1}^n (k_i+l_i))X(p;l_j-1). \quad (76)$$

The proof is by direct computation, using proposition 1. The first two relations are just a reformulation of the equalities in proposition 1. The next two relations follow immediately from the definition of $X(p;\boldsymbol{k},\boldsymbol{l})$ and the actions (54)-(55). The last two relations are the most difficult to verify. They follow from the action of the annihilation operators on each part of $X(p;\boldsymbol{k},\boldsymbol{l})$, using proposition 1, and then collecting the terms according to equal weight vectors $|p;\boldsymbol{k},\boldsymbol{l}\rangle$. This computation is long but straightforward, and will not be given in detail here. □

Note that all vectors $X(p;\boldsymbol{k},\boldsymbol{l})$ with $\sum_i(k_i+l_i)=p$ are annihilated by P.

In order to understand the linear (in)dependance of the newly introduced weight vectors $X(p;\boldsymbol{k},\boldsymbol{l})$, it is convenient to prove first an interesting lemma about the determinant and rank of a matrix. Let r be a positive integer, and consider the $2^r\times 2^r$ matrix A, where the rows and columns of A are labelled by the binary sequences $\boldsymbol{l}=(l_1,\ldots,l_r)$ ($l_i\in\{0,1\}$) of length r in reverse binary order. The elements $a_{\boldsymbol{l},\boldsymbol{l}'}$ of $A\equiv A(s;t_1,\ldots,t_r)$ are as follows

$$\begin{array}{l}
\text{if } \boldsymbol{l}=\boldsymbol{l}' \text{ then } a_{\boldsymbol{l},\boldsymbol{l}'}=s,\\
\text{if } \boldsymbol{l} \text{ and } \boldsymbol{l}' \text{ differ in only position } i \text{ then}\\
\qquad \text{if } l_i=0 \text{ then } a_{\boldsymbol{l},\boldsymbol{l}'}=-(-1)^{l_{i+1}+\cdots+l_r}t_i,\\
\qquad \text{if } l_i=1 \text{ then } a_{\boldsymbol{l},\boldsymbol{l}'}=(-1)^{l_i+\cdots+l_r}t_i,\\
\text{otherwise } a_{\boldsymbol{l},\boldsymbol{l}'}=0.
\end{array} \quad (77)$$

Herein, s and $t_1,\ldots,t_r$ are arbitrary real numbers or variables.

It is constructive to consider an example. For $r=2$, the binary sequences labelling the rows and columns of A are $(0,0),(1,0),(0,1),(1,1)$ (in this order); for $r=3$ the sequences are $(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)$. The matrices take

the following form :

$$A(s;t_1,t_2)=\begin{pmatrix} s & -t_1 & -t_2 & 0 \\ -1 & s & 0 & -t_2 \\ -1 & 0 & s & t_1 \\ 0 & -1 & 1 & s \end{pmatrix}, \tag{78}$$

$$A(s;t_1,t_2,t_3)=\begin{pmatrix} s & -t_1 & -t_2 & 0 & -t_3 & 0 & 0 & 0 \\ -1 & s & 0 & -t_2 & 0 & -t_3 & 0 & 0 \\ -1 & 0 & s & t_1 & 0 & 0 & -t_3 & 0 \\ 0 & -1 & 1 & s & 0 & 0 & 0 & -t_3 \\ -1 & 0 & 0 & 0 & s & t_1 & t_2 & 0 \\ 0 & -1 & 0 & 0 & 1 & s & 0 & t_2 \\ 0 & 0 & -1 & 0 & 1 & 0 & s & -t_1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & s \end{pmatrix} \tag{79}$$

Lemma 3 *The matrix $A(s;t_1,\dots,t_r)$ $(r>1)$ defined above satisfies :*

(a) the determinant is given by

$$\det A(s;t_1,\dots,t_r)=(s^2-\sum_{i=1}^{r}t_i)^{2^{r-1}}; \tag{80}$$

(b) if the elements t_i are positive real numbers such that $\sum_{i=1}^{r}t_i=s^2$ then $\operatorname{rank}(A(s;t_1,\dots,t_r))=2^{r-1}$.

(c)

$$A(s;t_1,\dots,t_r)\cdot A(-s;t_1,\dots,t_r)=(\sum_{i=1}^{r}t_i-s^2)I, \tag{81}$$

where I is the identity matrix of order 2^r.

Proof. The proof of (a) is by induction on r. Clearly it holds for $r=2$. By definition, the matrix $A(s;t_1,\dots,t_r)$ can be written in block form as

$$A(s;t_1,\dots,t_r)=\begin{pmatrix} A(s;t_1,\dots,t_{r-1}) & -t_rI \\ -I & -A(-s;t_1,\dots,t_{r-1}) \end{pmatrix}, \tag{82}$$

where I is the identity matrix of order 2^{r-1}. So by induction we have $\det(A(s;t_1,\dots,t_{r-1}))=(s^2-\sum_{i=1}^{r-1}t_i)^{2^{r-2}}$ and $\det(-A(-s;t_1,\dots,t_{r-1}))=((-s)^2-\sum_{i=1}^{r-1}t_i)^{2^{r-2}}$. Then it follows from (82) that

$$\begin{aligned}&\det(A(s;t_1,\dots,t_{r-1},0))=\\ &\det(A(s;t_1,\dots,t_{r-1}))\det(-A(-s;t_1,\dots,t_{r-1}))=(s^2-\sum_{i=1}^{r-1}t_i)^{2^{r-1}}.\end{aligned} \tag{83}$$

On the other hand, exchanging t_i and t_j in $A(s; t_1, \ldots, t_r)$ corresponds to a permutation of the rows and corresponding columns of $A(s; t_1, \ldots, t_r)$; a closer examination shows that the signature of such a permutation is positive. Thus $A(s; t_1, \ldots, t_r)$ is invariant for transpositions of the form $t_i \leftrightarrow t_j$. Therefore, $\det A(s; t_1, \ldots, t_r)$ is a symmetric polynomial in the elements t_i. Since the power sum symmetric functions form a basis of the ring of symmetric polynomials [33, p. 24], it follows that

$$\det A(s; t_1, \ldots, t_{r-1}, t_r) = \sum_{k=0}^{N} \sum_{\kappa \vdash k} c_\kappa(s) p_\kappa(t_1, \ldots, t_{r-1}, t_r), \tag{84}$$

where $N = 2^{r-1}$, κ is summed over all partitions of k, and p_κ is the multiplicative power sum function [33, p. 24]. Combining (84) with (83) gives

$$\det A(s; t_1, \ldots, t_{r-1}, 0) = \sum_{k=0}^{N} \sum_{\kappa \vdash k} c_\kappa(s) p_\kappa(t_1, \ldots, t_{r-1}) = (s^2 - p_1(t_1, \ldots, t_{r-1}))^N, \tag{85}$$

since $p_1(t_1, \ldots, t_{r-1}) = \sum_{i=1}^{r-1} t_i$. Thanks to the linear independence of the p_κ, the expansion of the factor to the Nth power in (85) fixes all the coefficients $c_\kappa(s)$. Substituting these back into (84), it follows that we must have

$$\det A(s; t_1, \ldots, t_{r-1}, t_r) = \sum_{k=0}^{N} \sum_{\kappa \vdash k} c_\kappa(s) p_\kappa(t_1, \ldots, t_{r-1}, t_r) = (s^2 - p_1(t_1, \ldots, t_{r-1}, t_r))^N, \tag{86}$$

which proves (a).

To prove (b), write $A(s; t_1, \ldots, t_r)$ as $A(s; t_1, \ldots, t_r) = sI - B$, where I is the identity matrix of order 2^r. Note that B is a matrix with elements similar to those of A but with zeros on the diagonal. Introducing a diagonal matrix D of order 2^r by

$$d_{\boldsymbol{l},\boldsymbol{l}} = \left(t_1^{l_1} t_2^{l_2} \cdots t_r^{l_r}\right)^{1/2}, \tag{87}$$

it is easy to see that DBD^{-1} is a real and symmetric matrix. For such matrices, all eigenvalues are real, with geometric multiplicity equal to the algebraic multiplicity. Thus this also holds for the eigenvalues of B. The characteristic equation of B is

$$\det(B - \lambda I) = \det(A(\lambda; t_1, \ldots, t_r)) = (\lambda^2 - \sum_{i=1}^{r} t_i)^{2^{r-1}} = (\lambda - \mu)^{2^{r-1}} (\lambda + \mu)^{2^{r-1}}, \tag{88}$$

where $\mu = +\sqrt{\sum_{i=1}^{r} t_i}$. So for the eigenvalue μ the geometric multiplicity is 2^{r-1}. Since the geometric multiplicity is also equal to $2^r - \operatorname{rank}(B - \mu I)$, it follows that $\operatorname{rank}(B - \mu I) = 2^r - 2^{r-1} = 2^{r-1}$, or $\operatorname{rank}(A(\mu; t_1 \ldots, t_r)) = 2^{r-1}$, implying the statement (b).

To prove (c), consider the element $c_{\boldsymbol{l},\boldsymbol{l}'}$ in the multiplication of $A(s; t_1, \ldots, t_r)$ with $A(-s; t_1, \ldots, t_r)$. From the definition (77) it follows immediately that $c_{\boldsymbol{l},\boldsymbol{l}'} = 0$ if $\boldsymbol{l} \neq \boldsymbol{l}'$, and that $c_{\boldsymbol{l},\boldsymbol{l}} = \sum_i t_i - s^2$. $\square$

This lemma can now be used to determine the linear (in)dependence of the weight vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$. Consider a weight $\lambda_m = (p - \sum_i m_i, m_1, \ldots, m_n)$, with all $m_i \geq 0$. Then the multiplicity of λ_m in $\bar{V}_p$, or equivalently the dimension of the weight space $\bar{V}_p(\lambda_m)$, is given by

$$d_m = \dim \bar{V}_p(\lambda_m) = 2^{\gamma(m_1)+\cdots+\gamma(m_n)} = 2^r, \tag{89}$$

where r is the number of nonzero m_i's. A basis of $\bar{V}_p(\lambda_m)$ is given by the set of vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$ with every $k_i + l_i = m_i$ (or $\boldsymbol{k} + \boldsymbol{l} = \boldsymbol{m}$). One can consider another set of d_m vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$ with $\boldsymbol{k} + \boldsymbol{l} = \boldsymbol{m}$. The coefficient matrix of the vectors $(-1)^{l_1+\cdots+l_n} X(p; \boldsymbol{k}, \boldsymbol{l})$ expressed in terms of the vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$ coincides with the matrix $A(\sqrt{p}; t_1, \ldots, t_r)$ defined in Lemma 3, with t_i corresponding to the nonzero m_i's. Thus the determinant of this matrix is

$$(p - \sum_{i=1}^{n} m_i)^{d_m/2}. \tag{90}$$

In other words, if $\sum_{i=1}^n m_i \neq p$, then the coefficient matrix is nonsingular, and the d_m vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$ with $\boldsymbol{k} + \boldsymbol{l} = \boldsymbol{m}$ form a basis for $\bar{V}_p(\lambda_m)$. When $\sum_{i=1}^n m_i = p$, it follows from Lemma 3(b) that the span of the d_m vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$ with $\boldsymbol{k} + \boldsymbol{l} = \boldsymbol{m}$ is a subspace of $\bar{V}_p(\lambda_m)$ of dimension $d_m/2$.

7 The simple module V_p

Denote by M_p the maximal $G = q(n+1)$ submodule of $\bar{V}_p$ (different from $\bar{V}_p$ itself). Then the quotient module $V_p = \bar{V}_p / M_p$ is an irreducible (or simple) $q(n+1)$ module. In this section we shall show that V_p is finite dimensional, and give its weight structure, character, and dimension. By definition, V_p and M_p are weight modules, and

$$v \in M_p \Leftrightarrow v_0 \notin U(G)v. \tag{91}$$

For the weight vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$ or $X(p; \boldsymbol{k}, \boldsymbol{l})$ it will be useful to refer to the quantity $\sum_{i=1}^n (k_i + l_i) = \sum_i m_i$ as the *level* of the vector (or of the corresponding weight).

Proposition 4 *The weight vectors v of $\bar{V}_p$ satisfy the following :*

(a) if the level of v is greater than p then $v \in M_p$;

(b) if the level of v is less than p then $v \notin M_p$, and denoting the vectors of the quotient module by their representatives in $\bar{V}_p$ we can write $v \in V_p$;

(c) if the level of v is equal to p, consider its weight $\lambda_m = (0, m_1, \ldots, m_n)$. With $d_m = \dim(\bar{V}_p(\lambda_m))$, we have that $\dim M_p(\lambda_m) = \dim(V_p(\lambda_m)) = d_m/2$. Moreover, the vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$ of level p with $k_i + l_i = m_i$ span $M_p(\lambda_m)$.

Proof. Consider a fixed weight λ_m, and the corresponding weight vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$ and $X(p; \boldsymbol{k}, \boldsymbol{l})$. If the level of λ_m is less than p, then the vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$ form a basis for $\bar{V}_p(\lambda_m)$ (see end of section 6). By (75) and (76),

$$(b_1^-)^{k_1}(f_1^-)^{l_1} \cdots (b_n^-)^{k_n}(f_n^-)^{l_n} X(p; \boldsymbol{k}, \boldsymbol{l})$$

is equal to a nonzero constant times v_0. Thus $v_0 \in U(G)X(p; \boldsymbol{k}, \boldsymbol{l})$, in other words all vectors of weight λ_m are not in M_p.

Let the level of v be greater than p. Applying b_j^- or f_j^- reduces the level by one. But for the vectors at level $p+1$ it follows from (71) and (72) that the action of b_j^- or f_j^- yields only vectors of the form $X(p; \boldsymbol{k}, \boldsymbol{l})$ at level p, and all these vectors themselves are annihilated by b_j^- and f_j^-. Thus one deduces that v_0 cannot belong to $U(G)v$ if v has level greater than p.

Finally, consider a weight $\lambda_m = (0, m_1, \ldots, m_n)$ of level p. The vectors $X(p; \boldsymbol{k}, \boldsymbol{l})$ with $k_i + l_i = m_i$ are all annihilated by b_j^- and f_j^-, so it follows that they belong to M_p. In this case, we know (see end Section 6) that these vectors span a subspace of dimension $d_m/2$. Thus $\dim M_p(\lambda_m) \geq d_m/2$, and we still need to show that the dimension of $M_p(\lambda_m)$ does not exceed $d_m/2$. The space $\bar{V}_p(\lambda_m)$, of dimension d_m, is spanned by the d_m vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$ with $k_i + l_i = m_i$. Assume that $m_n \neq 0$ (the same argument works for another $m_i \neq 0$). Consider the sets

$$
\begin{aligned}
S_0 &= \{X(p; k_1, l_1, \ldots, k_{n-1}, l_{n-1}, m_n, 0) | k_i + l_i = m_i\}, &(92)\\
S_1 &= \{|p; k_1, l_1, \ldots, k_{n-1}, l_{n-1}, m_n - 1, 1\rangle | k_i + l_i = m_i\}, &(93)\\
S &= S_0 \cup S_1. &(94)
\end{aligned}
$$

Clearly, $\#S_0 = \#S_1 = d_m/2$. The vectors in S_1 are obviously linearly independent. By considering the coefficient matrix of the vectors of S_0 in terms of the d_m vectors $|p; \boldsymbol{k}, \boldsymbol{l}\rangle$, and using Lemma 3, it follows that the vectors of S_0 are also linearly independent, and furthermore that the vectors of S are linearly independent. Thus S constitutes a basis for $\bar{V}_p(\lambda_m)$. The elements of S_0 all belong to $M_p(\lambda_m)$. On the other hand, span(S_1) contains no vectors that are annihilated by P. Indeed, consider a linear combination c of the vectors

in S_1, and express that c is annihilated by b_n^-. Using (71), and linear independence of the vectors appearing in $b_n^- c$, it follows that $b_n^- c = 0$ only if all the coefficients in the linear combination c are zero. One can now deduce that no linear combination of S_1 can yield a vector of $M_p(\lambda_m)$. This shows that $\dim M_p(\lambda_m) = d_m/2$, hence $\dim V_p(\lambda_m) = d_m/2$. $\square$

It can be verified that M_p, which is a $q(n+1)$ module, is generated by one vector w as a $q(n+1)$ module (otherwise said : $\bar{V}_p$ containes one $q(n+1)$ highest weight singular vector w). In our notation, this vector w is equal to $w = X(p; p, 0, \ldots, 0, 0, \ldots 0)$.

This proposition gives us explicitly the vectors of M_p. Hence it also gives the (representatives of) the vectors of $V_p = \bar{V}_p/M_p$. In particular, V_p is finite dimensional, and the weight structure of V_p can be deduced. For a weight $\lambda_m = (p - \sum_i m_i, m_1, \ldots, m_n)$ (with all $m_i \geq 0$), let r be the number of nonzero m_i's. Then the multiplicity of λ_m is 2^r if the level of λ_m is less than p, and $2^r/2 = 2^{r-1}$ if the level of λ_m is equal to p (and, of course, zero of the level is larger than p).

Once the weight structure is known, it is possible to write down the character and dimension of V_p. To do this, it will be useful to first determine the decomposition of V_p with respect to the subalgebra $gl(n+1) \subset G$. Since V_p is finite dimensional, it will decompose into a direct sum of simple finite dimensional $gl(n+1)$ modules. This decomposition can be derived from the weight structure. The highest weight is $(p, 0, \ldots, 0)$, with multiplicity 1. So the $gl(n+1)$ module with highest weight $(p, 0, \ldots, 0)$ is a component of the decomposition. Subtracting the (known) weights of this $gl(n+1)$ module from the set of weights of V_p, leaves $(p-1, 1, 0, \ldots, 0)$ as the next highest weight, also with multiplicity one. Then we go on : first subtract all weights (including multiplicities) of the $gl(n+1)$ module labelled by $(p-1, 1, 0, \ldots, 0)$; then determine the highest weight of the remaining ones, etc. Finally, one obtains :

Proposition 5 *The decomposition of the $q(n+1)$ module V_p into $gl(n+1)$ modules (with each $gl(n+1)$ module characterized by its highest weight) is as follows :*

$$V_p \to (p, 0, \ldots, 0) \oplus (p-1, 1, 0, \ldots, 0) \oplus (p-2, 1, 1, 0, \ldots, 0) \oplus \cdots \oplus (p-n, 1, 1, \ldots, 1). \quad (95)$$

The dimension of V_p is given by :

$$\dim V_p = \sum_{i=0}^{n} \binom{p-1}{i} \binom{p+n-i}{n-i}. \quad (96)$$

Proof. The decomposition follows from the known weight structure determined in Proposition 4. From the known dimension formula (e.g. [34, §4]) of simple $gl(n+1)$ modules, (96) follows. $\square$

The (formal) character of a G module V is defined as usual :

$$\mathrm{ch}\, V = \sum_{\lambda} \dim V(\lambda) x_0^{\lambda_0} \cdots x_n^{\lambda_n}, \tag{97}$$

where $\dim V(\lambda)$ is the multiplicity of a weight λ in V, and the x_i can be considered as formal variables. The character of V_p follows from (95), using the (known) characters of $gl(n+1)$ modules. For a $gl(n+1)$ module with highest weight λ, the character is equal to the Schur function $s_\lambda(x_0, \ldots, x_n)$. In this case, the highest weights appearing in the decomposition are of a special form; in fact, they are of Frobenius form $(p-1-i|i)$ [33, p. 3]. Since in such a case the character is given by [33, p. 47] :

$$s_{(a|b)} = h_{a+1}e_b - h_{a+2}e_{b-1} + \cdots + (-1)^b h_{a+b+1}, \tag{98}$$

where e_r (resp. h_r) is the rth elementary (resp. complete) symmetric function in the x_i, it follows that

$$\mathrm{ch}\, V_p = h_{p-n}e_n + h_{p-n-2}e_{n-2} + \cdots, \tag{99}$$

ending with h_p if n is even and with $h_{p-1}e_1$ if n is odd.

To have the interpretation of V_p as a Fock space of $q(n+1)$, we still need to show that the Hermitian form is positive definite.

Proposition 6 *The Hermitian form on V_p, induced by (25), is positive definite.*

Proof. It is clear from (25) and (91) that the Hermitian form is zero on M_p, so (25) induces indeed a Hermitian form on $V_p = \bar{V}_p/M_p$. It also follows from (25) that $\langle v|w\rangle = 0$ if the weight of v and w is different. So it is sufficient to study the behaviour of the Hermitian form on a weight space $V_p(\lambda_m)$ only. Let λ_m be fixed, and assume that the level of λ_m is less than p ($\sum m_i < p$) and that all m_i are nonzero (the proof has be slightly changed in the remaining cases since according to Proposition 4 a different basis must be chosen; but in this different basis, it leads essentially to the same computation). A basis for $V_p(\lambda_m)$ is given by the vectors $|p;\boldsymbol{k},\boldsymbol{l}\rangle$ with $\boldsymbol{k}+\boldsymbol{l}=\boldsymbol{m}$. Thus we have to show that the matrix H with matrix elements

$$H_{\boldsymbol{l},\boldsymbol{l}'} = \langle\ |p;\boldsymbol{k},\boldsymbol{l}\rangle \mid |p;\boldsymbol{k}',\boldsymbol{l}'\rangle\ \rangle \tag{100}$$

is positive definite. By (50), $H_{\boldsymbol{l},\boldsymbol{l}'}$ is equal to the coefficient of v_0 in

$$(f_n^-)^{l_n}(b_n^-)^{k_n} \cdots (f_1^-)^{l_1}(b_1^-)^{k_1}|p;\boldsymbol{k}',\boldsymbol{l}'\rangle. \tag{101}$$

The idea is now as follows :

- The coefficient matrix of the vectors $(-1)^{l_1+\cdots+l_n} X(p;\boldsymbol{k},\boldsymbol{l})$ expressed in terms of the vectors $|p;\boldsymbol{k},\boldsymbol{l}\rangle$ is given by $A = A(\sqrt{p}; m_1,\ldots,m_n)$. Thus the coefficient matrix of the vectors $|p;\boldsymbol{k},\boldsymbol{l}\rangle$ expressed in terms of $(-1)^{l_1+\cdots+l_n} X(p;\boldsymbol{k},\boldsymbol{l})$ is given by A^{-1}.

- The action of $(f_n^-)^{l_n}(b_n^-)^{k_n}\cdots(f_1^-)^{l_1}(b_1^-)^{k_1}$ on a vector of the form $(-1)^{l'_1+\cdots+l'_n} X(p;\boldsymbol{k'},\boldsymbol{l'})$ is diagonal, and determined by (75) and (76).

This leads to

$$H_{\boldsymbol{l},\boldsymbol{l'}} = d(\boldsymbol{k},\boldsymbol{l})(A^{-1})_{\boldsymbol{l'},\boldsymbol{l}}, \tag{102}$$

where

$$d(\boldsymbol{k},\boldsymbol{l}) = k_1!k_2!\cdots k_n!(p-1)(p-2)\cdots(p-\sum m_i).$$

It follows that $H = cD^{-1}A^{-T}$, with c a positive constant, A^{-T} the transpose of A^{-1}, and D a diagonal matrix with elements $D_{\boldsymbol{l},\boldsymbol{l}} = m_1^{l_1}\cdots m_n^{l_n}$. But H (being symmetric) is positive definite if and only if $D^{1/2}HD^{1/2}$ is positive definite (e.g. by the Cholesky decomposition). Now $D^{1/2}HD^{1/2} = cD^{-1/2}A^{-T}D^{1/2}$; this last matrix is positive definite if all its eigenvalues are positive. From the proof of Lemma 3(b) (and A^{-1} determined by Lemma 3(c)),

$$\det(D^{-1/2}A^{-T}D^{1/2} - \lambda I) = \left((\sqrt{p}-\lambda)^2 - \sum_i m_i \right),$$

so the eigenvalues are $\lambda = \sqrt{p} \pm \sqrt{\sum_i m_i}$, which are indeed positive since $\sum_i m_i < p$. $\square$

8 Conclusion

We have given a description of the Lie superalgebra $q(n+1)$ in terms of creation operators b_i^+, f_i^+ and annihilation operators b_i^-, f_i^- $(i = 1,\ldots,n)$. The quadratic relations (9) and the triple supercommutation relations (10) and (11) determine the Lie superalgebra $sq(n+1)$ completely. The operators $b_i^\pm$ satisfy the relations of A-statistics, and the operators $f_i^\pm$ the relations of A-superstatistics. The combined relations (9)-(11) can be seen as a unification of A-statistics and A-superstatistics.

We have shown that $q(n+1)$ has an interesting class of irreducible representations V_p, defined as a quotient module of an induced module $\bar{V}_p$. For p a positive integer, these representations V_p are finite-dimensional, with a unique highest weight (of multiplicity 1). The Hermitian form that is consistent with the natural adjoint operation on $q(n+1)$ is shown to be positive definite on V_p. For $q(2)$ these representations are “dispin”, since they decompose into the sum of two irreducible $gl(2)$ representation in the decomposition

$q(2) \supset gl(2)$. Also in the general case, the decomposition of V_p with respect to $q(n+1) \supset gl(n+1)$ is determined, through the weight structure of V_p. Thus a character and dimension formula for V_p is given.

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